

ON DIFFEOMORPHISMS OVER NON-ORIENTABLE SURFACES STANDARDLY EMBEDDED IN THE 4-SPHERE

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ABSTRACT. For a non-orientable closed surface standardly embedded in the 4-sphere, a diffeomorphism over this surface is extendable if and only if this diffeomorphism preserves the Guillou-Marin quadratic form of this embedded surface.

1. INTRODUCTION

Let S be a closed surface and e be a smooth embedding of S into S^4 . A diffeomorphism ϕ over S is *e-extendable* if there is an orientation preserving diffeomorphism Φ of S^4 such that $\Phi|_S = \phi$. The natural problem to ask is: *Find a necessary and sufficient condition for a diffeomorphism over S to be e-extendable?*

For some special embeddings of closed surfaces in 4-manifolds, we have answers to the above problem (for example, [12], [4], [6]). An embedding e of the orientable surface Σ_g into S^4 is called standard if $e(\Sigma_g)$ is the boundary of 3-dimensional handlebody embedded in S^4 . In [12] and [4], we showed:

Theorem 1.1 ([12] ($g = 1$), [4] ($g \geq 2$)). *Let Σ_g be standardly embedded in S^4 . An orientation preserving diffeomorphism ϕ over the Σ_g is extendable to S^4 if and only if ϕ preserves the Rokhlin quadratic form of the Σ_g standardly embedded in S^4 .*

In this paper, we consider the same kind of problem for non-orientable surfaces embedded in S^4 . Let N_g be a connected non-orientable surface constructed from g projective planes by connected sum. We call N_g *the closed non-orientable surface of genus g* . Let $S^3 \times [-1, 1]$ be a closed tubular neighborhood of the equator S^3 in S^4 . Then $S^4 - S^3 \times (-1, 1)$ consists of two 4-balls. An embedding $os : N_g \hookrightarrow S^4$ is *o-standard* if $os(N_g) \subset S^3 \times [-1, 1]$ and as shown in Figure 1. The main result of this paper is:

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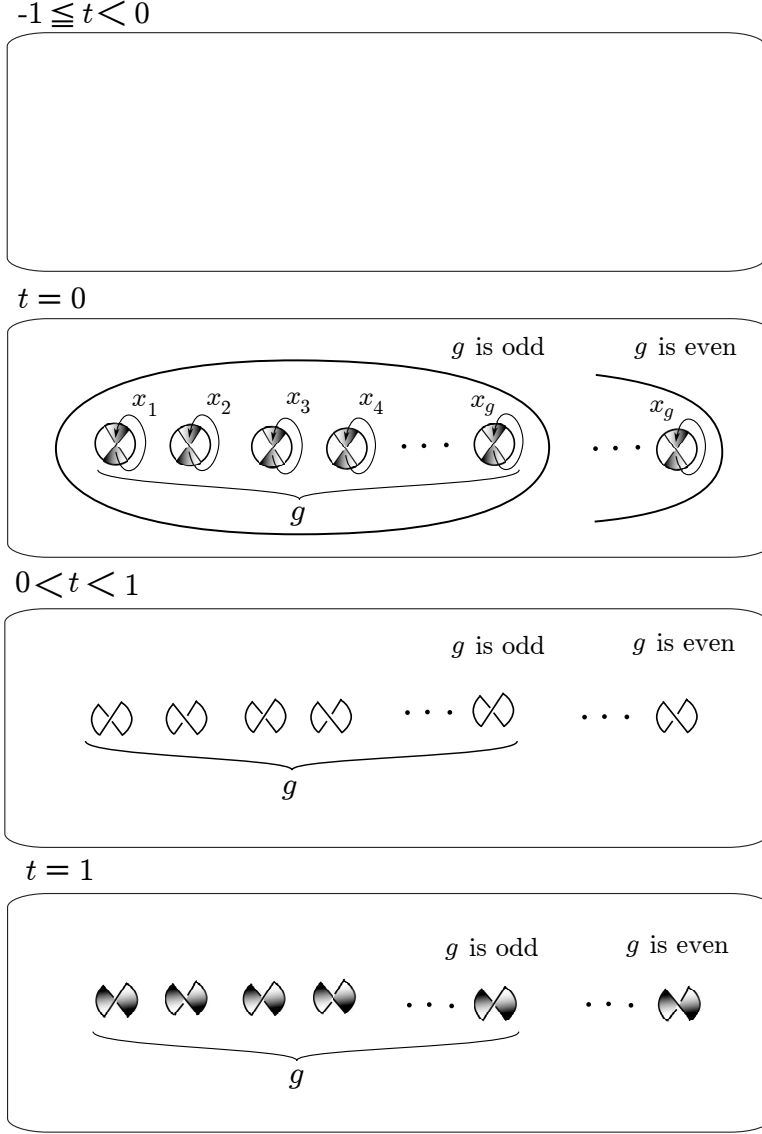


FIGURE 1. The motion picture of the o -standard embedding of N_g into S^4 .

Theorem 1.2. *The diffeomorphism ϕ over N_g is os -extendable if and only if ϕ preserves the Guillou-Marin quadratic form of the N_g o -standardly embedded in S^4 .*

2. GUILLOU-MARIN QUADRATIC FORM

For a smooth embedding e of the closed non-orientable surface N_g of genus g into S^4 , Guillou and Marin ([3] see also [11]) defined a quadratic form $q_e : H_1(N_g; \mathbb{Z}_2) \rightarrow \mathbb{Z}_4$ as follows: Let C be an immersed circle on N_g , and D be a connected orientable

surface immersed in S^4 such that $\partial D = C$, and D is not tangent to N_g . Let ν_D be the normal bundle of D , then $\nu_D|_C$ is a solid torus with a trivialization induced from any trivialization of ν_D . Let $N_{N_g}(C)$ be the tubular neighborhood of C in N_g , then $N_{N_g}(C)$ is an twisted annulus or a Möbius band in $\nu_D|_C$. We denote by $n(D)$ the number of right hand half-twists of $N_{N_g}(C)$ with respect to the trivialization of $\nu_D|_C$. Let $D \cdot F$ be the mod-2 intersection number between D and F , $Self(C)$ be the mod-2 double points number of C , and $2 \times$ be an injection $\mathbb{Z}_2 \rightarrow \mathbb{Z}_4$ defined by $2 \times [n]_2 = [2n]_4$. Then the number $n(D) + 2 \times D \cdot F + 2 \times Self(C) \pmod{4}$ depend only on the mod-2 homology class $[C]$ of C . Hence, we define

$$q_e([C]) = n(D) + 2 \times D \cdot F + 2 \times Self(C) \pmod{4}.$$

This map q_e is called Guillou-Marin *quadratic form*, since q_e satisfies

$$q_e(x + y) = q_e(x) + q_e(y) + 2 \times (x \cdot y)_2,$$

where $(x \cdot y)_2$ means the mod-2 intersection number between x and y . For example, $q_{os}(x_{2i-1}) = +1, q_{os}(x_{2i}) = -1$ for the basis $\{x_1, \dots, x_g\}$ of $H_1(N_g; \mathbb{Z}_2)$ shown in Figure 1. This quadratic form q_e is a non-orientable analogy of Rokhlin quadratic form.

A diffeomorphism ϕ over N_g is *e-extendable* if there is an orientation preserving diffeomorphism Φ of S^4 such that the following diagram is commutative,

$$\begin{array}{ccc} N_g & \xrightarrow{e} & S^4 \\ \phi \downarrow & & \downarrow \Phi \\ N_g & \xrightarrow{e} & S^4. \end{array}$$

If the diffeomorphisms ϕ_1 over N_g is *e-extendable*, and ϕ_1 is isotopic to ϕ_2 , then ϕ_2 is *e-extendable*. Therefore, *e-extendability* is a property about isotopy classes of diffeomorphisms over N_g . The group $\mathcal{M}(N_g)$ of isotopy classes of all diffeomorphisms over N_g is called *the mapping class group of N_g* . An element ϕ of $\mathcal{M}(N_g)$ is *e-extendable* if there is an *e-extendable* representative of ϕ . By the definition of q_e , we can see that if $\phi \in \mathcal{M}(N_g)$ is *e-extendable* then ϕ preserves q_e , i.e. $q_e(\phi_*(x)) = q_e(x)$ for every $x \in H_1(N_g; \mathbb{Z}_2)$. What we would like to know is whether $\phi \in \mathcal{M}(N_g)$ is *e-extendable* when ϕ preserves q_e . The answer to this problem would be depend on the embedding e . In this paper, we consider the case where e is the *o*-standard embedding.

3. GENERATORS FOR $\mathcal{M}(N_g)$

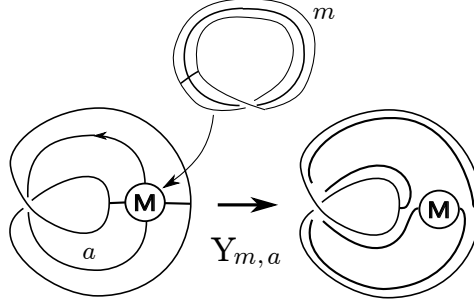


FIGURE 2. M with circle indicates a place where to attach a Möbius band

A simple closed curve c on N_g is an A-circle (resp. an M-circle), if the tubular neighborhood of c is an annulus (resp. a Möbius band). We denote by t_c the Dehn twist about an A-circle c on N_g . In each figure, we indicate the direction of a Dehn twist by an arrow. Lickorish [9, 10] showed that Dehn twists and Y -homeomorphisms generate $\mathcal{M}(N_g)$. We review the definition of Y -homeomorphism. Let m be an M-circle and a be an oriented A-circle in N_g such that m and a transversely intersect in one point. Let $K \subset N_g$ be a regular neighborhood of $m \cup a$, which is a union of the tubular neighborhoods of m and a and then is homeomorphic to the Klein bottle with a hole. Let M be a regular neighborhood of m . We denote by $Y_{m,a}$ a homeomorphism over N_g which is described as the result of pushing M once along a keeping the boundary of K fixed (see Figure 2). We call $Y_{m,a}$ a Y -homeomorphism. Szepietowski [14] showed an interesting results on the proper subgroup of $\mathcal{M}(N_g)$ generated by all Y -homeomorphisms.

Theorem 3.1 ([14]). $\Gamma_2(N_g) = \{\phi \in \mathcal{M}(N_g) \mid \phi_* : H_1(N_g; \mathbb{Z}_2) \rightarrow H_1(N_g; \mathbb{Z}_2) = id\}$ is generated by Y -homeomorphisms.

Chillingworth showed that $\mathcal{M}(N_g)$ is finitely generated.

Theorem 3.2 ([2]). Let a_1, \dots, a_{g-1}, b_j ($1 \leq j \leq 2/g$) and m_{g-1} be circles shown in Figure 3. Then $t_{a_1}, \dots, t_{a_{g-1}}, t_{b_j}$ ($1 \leq j \leq 2/g$), $Y_{m_{g-1}, a_{g-1}}$ generate $\mathcal{M}(N_g)$.

Remark 3.3. If $g = 1$, then $\mathcal{M}(N_1)$ is trivial, hence Theorem 1.2 is valid. From here to the end of this paper, we assume $g \geq 2$.

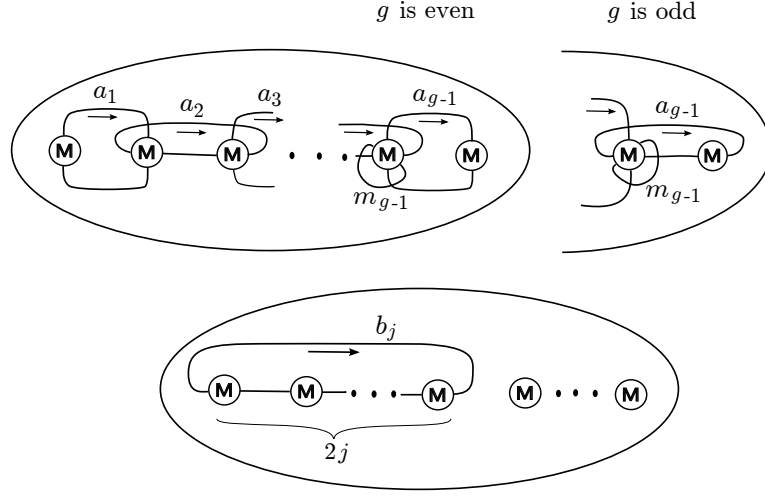


FIGURE 3. Chillingworth's generators for $\mathcal{M}(N_g)$.

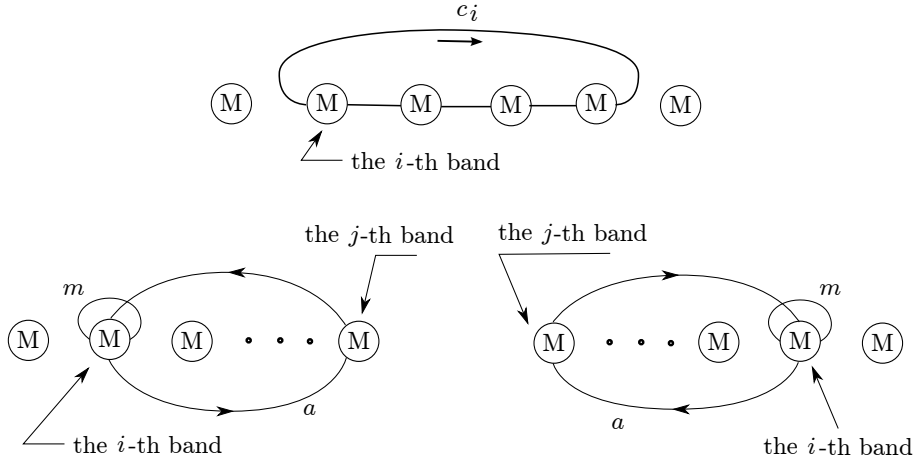


FIGURE 4. The circle c_i and the Y-homeomorphism $Y_{i,j}$.

The i -th band is the Möbius band on which the circle x_i in Figure 1 goes across. Let c_i ($i = 1, \dots, g-3$) be a simple closed curve shown in the top of Figure 4. Let $i, j = 1, \dots, g$ such that $i \neq j$. When $i < j$ (resp. $i > j$), we define $Y_{i,j} = Y_{m,a}$, where m and a are as shown in the left bottom (resp. the right bottom) of Figure 4. Let \mathcal{YS}_g be the subgroup of $\mathcal{M}(N_g)$ generated by all $Y_{i,j}$.

Lemma 3.4. \mathcal{YS}_g and $t_{a_1}, \dots, t_{a_{g-1}}, t_{c_1}, \dots, t_{c_{g-3}}$ generate $\mathcal{M}(N_g)$.

Proof. It suffices to show that t_{b_j} is a product of $t_{a_1}, \dots, t_{a_{g-1}}, t_{c_1}, \dots, t_{c_{g-3}}$. When $j = 2$, $t_{b_2} = t_{c_1}$. When $j \geq 3$, by the lantern relation which was discovered by

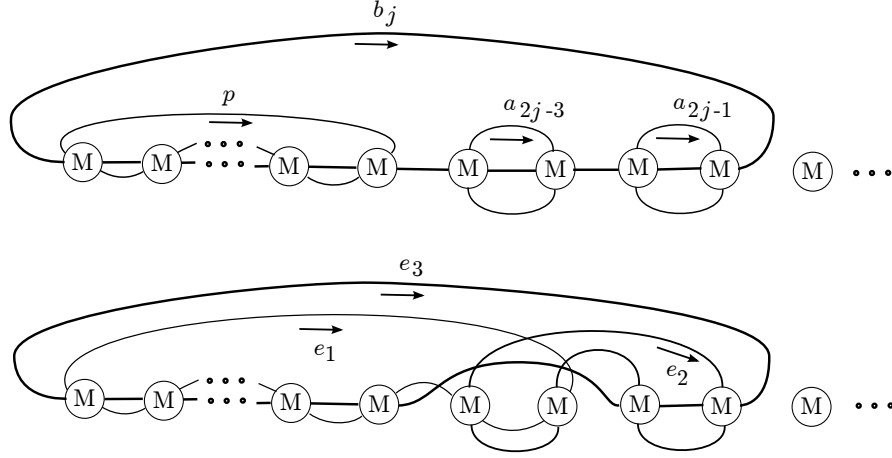


FIGURE 5. When $j = 3$, we drop the 2-nd and 3-rd M with circles.

Dehn and rediscovered by Johnson [7], $t_{e_1}t_{e_2}t_{e_3} = t_pt_{a_{2j-3}}t_{a_{2j-1}}t_{b_j}$, where p , e_1, e_2 and e_3 are circles shown in Figure 5, hence $t_{b_j} = (t_pt_{a_{2j-3}}t_{a_{2j-1}})^{-1}t_{e_1}t_{e_2}t_{e_3}$. Since $e_1 = t_{a_{2j-2}}t_{a_{2j-3}}t_{a_{2j-1}}t_{a_{2j-2}}(e_3)$, we see $t_{e_3} = (t_{a_{2j-2}}t_{a_{2j-3}}t_{a_{2j-1}}t_{a_{2j-2}})^{-1}t_{e_1}t_{a_{2j-2}}t_{a_{2j-3}}t_{a_{2j-1}}t_{a_{2j-2}}$. If $j = 3$, then $e_1 = c_1$, $e_2 = c_3$ and $p = a_1$. Therefore, t_{b_3} is a product of $t_{a_1}, \dots, t_{a_{g-1}}, t_{c_1}, \dots, t_{c_{g-3}}$. If $j \geq 4$, then $e_1 = b_{j-1}$, $e_2 = c_{2j-3}$, and $p = b_{j-2}$. By the induction on j , we see that t_{b_j} is a product of $t_{a_1}, \dots, t_{a_{g-1}}, t_{c_1}, \dots, t_{c_{g-3}}$. \square

Remark 3.5. If $g = 2, 3$, then this lemma reads \mathcal{YS}_g and $t_{a_1}, \dots, t_{a_{g-1}}$ generate $\mathcal{M}(N_g)$.

4. GENERATORS FOR SUBGROUP OF $\mathcal{M}(N_g)$ PRESERVING q_{os}

In this section, we find a finite system of generators for

$$\mathcal{N}_g(q_{os}) = \{\phi \in \mathcal{M}(N_g) \mid q_{os}(\phi_*(x)) = q_{os}(x) \text{ for every } x \in H_1(N_g; \mathbb{Z}_2)\}$$

and prove the main theorem (Theorem 1.2) of this paper.

We introduce a group

$$\mathcal{O}_g(q_{os}) = \{A \in \text{Aut}(H_1(N_g; \mathbb{Z}_2)) \mid q_{os}(A(x)) = q_{os}(x) \text{ for every } x \in H_1(N_g; \mathbb{Z}_2)\}.$$

Then we have a natural short exact sequence

$$(1) \quad 0 \rightarrow \Gamma_2(N_g) \rightarrow \mathcal{N}_g(q_{os}) \rightarrow \mathcal{O}_g(q_{os}) \rightarrow 0.$$

Since $\Gamma_2(N_g)$ is a finite index subgroup of $\mathcal{M}(N_g)$ and $\mathcal{O}_g(q_{os})$ is a finite group, there exists a finite system of generators for $\mathcal{N}_g(q_{os})$. We find a system of generators explicitly.

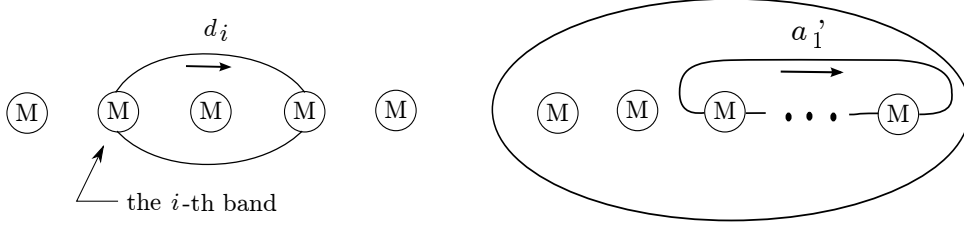


FIGURE 6. The circle d_i .

Theorem 4.1. $\mathcal{N}_g(q_{os})$ is generated by \mathcal{YS}_g , $t_{a_1}^2, \dots, t_{a_{g-1}}^2$, $t_{c_1}^2, \dots, t_{c_{g-3}}^2$, $t_{d_1}, \dots, t_{d_{g-2}}$, and $t_{a_1}t_{a_3}t_{c_1}, \dots, t_{a_{g-3}}t_{a_{g-1}}t_{c_{g-3}}$, where d_i is illustrated in Figure 6.

Remark 4.2. If $g = 2$, this theorem reads $\mathcal{N}_2(q_{os})$ is generated by \mathcal{YS}_2 and $t_{a_1}^2 = id_{N_2}$. If $g = 3$, this theorem reads $\mathcal{N}_3(q_{os})$ is generated by \mathcal{YS}_3 , $t_{a_1}^2, t_{a_2}^2$ and t_{d_1} .

Proof of Theorem 1.2. By the definition of q_{os} , if a diffeomorphism ϕ over N_g is os -extendable then ϕ preserves q_{os} .

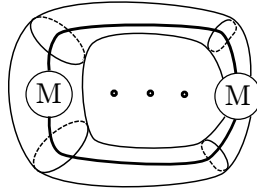


FIGURE 7. Sliding the left Möbius band along this tube is an extension of $Y_{i,j}$.

Conversely, we assume that ϕ preserves q_{os} . Then ϕ is an element of $\mathcal{N}_g(q_{os})$. Therefore, if each generator of $\mathcal{N}_g(q_{os})$ is os -extendable then ϕ is os -extendable. Since a sliding of a Möbius band along the tube illustrated in Figure 7 is an extension of $Y_{i,j}$, $Y_{i,j}$ is os -extendable, hence every element of \mathcal{YS}_g is os -extendable. Since the regular neighborhoods of a_i and c_i are annuli trivially embedded in the equator S^3 of S^4 , $t_{a_i}^2$, $t_{c_i}^2$ are os -extendable by the same argument as in the introduction of [5]. Since the regular neighborhoods of d_i is a Hopf band embedded in the equator S^3 of S^4 , t_{d_i} are

os-extendable by the same argument as the proof of [5, Proposition 2.1]. Finally, by using the same argument as showing the extendability of “ $C_1C_3C_5$ ” in the proof of [4, Lemma 2.2], we show that $t_{a_i}t_{a_{i+2}}t_{c_i}$ is os-extendable. \square

For $a \in H_1(N_g; \mathbb{Z}_2)$, we define the *transvection* $T_a : H_1(N_g; \mathbb{Z}_2) \rightarrow H_1(N_g; \mathbb{Z}_2)$ about a by $T_a(x) = x + (x \cdot a)_2 a$, where $(\cdot)_2$ means the mod-2 intersection form. We remark that if l is a simple closed curve on N_g such that $[l] = a \in H_1(N_g; \mathbb{Z}_2)$, then $(t_l)_* = T_a$. Since $T_a^2 = id_{H_1(N_g; \mathbb{Z}_2)}$, $t_l^2 \in \mathcal{N}_g(q_{os})$ for every simple closed curve l on N_g . For $a \in H_1(N_g; \mathbb{Z}_2)$ with $q_{os}(a) = 1$, T_a preserves q_{os} , hence $t_{d_i} \in \mathcal{N}_g(q_{os})$. For $a, b \in H_1(N_g; \mathbb{Z}_2)$ with $q_{os}(a) = q_{os}(b) = q_{os}(a+b) = 0$, $T_a T_b T_{a+b}$ preserves q_{os} , hence $t_{a_i} t_{a_{i+2}} t_{c_i}$ are elements of $\mathcal{N}_g(q_{os})$. Since $(Y_{i,j})_* = id_{H_1(N_g; \mathbb{Z}_2)}$, $\mathcal{YS}_g \subset \mathcal{N}_g(q_{os})$. Therefore, in order to prove Theorem 4.1, we should see that every element of $\mathcal{N}_g(q_{os})$ is a product of these elements.

4.1. Short-leg Y-homeomorphisms. For a Y-homeomorphism $Y_{m,a}$, we call m the *leg* of $Y_{m,a}$ and a the *arm* of $Y_{m,a}$. A Y-homeomorphism is called a *short-leg* Y-homeomorphism, if its leg is one of x_1, \dots, x_g illustrated in Figure 1.

Lemma 4.3. *Every short-leg Y-homeomorphism is an element of \mathcal{YS}_g .*

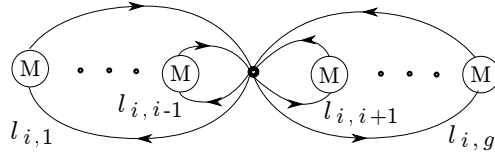


FIGURE 8. The generators for $\pi_1(N_g, p_0)$.

Proof. We review the *crosscap pushing map* defined in [15]. Fix $p_0 \in N_{g-1}$ and define $\mathcal{M}(N_{g-1}, p_0)$ be the group of isotopy classes of diffeomorphisms over N_{g-1} preserving p_0 . Let U be a 2-disk embedded in N_{g-1} such that the center of U is p_0 . We define a homomorphism j from $\pi_1(N_{g-1}, p_0)$ to $\mathcal{M}(N_{g-1}, p_0)$ such that, for a loop γ in N_g based at x_0 and an element $[\gamma] \in \pi_1(N_{g-1}, x_0)$, $j([\gamma])$ is a diffeomorphism over N_g obtained as the effect of pushing p_0 once along γ . This homomorphism j is in a non-orientable analogy of the Birman exact sequence [1]. We define a homomorphism φ from $\mathcal{M}(N_{g-1}, p_0)$ to $\mathcal{M}(N_g)$ as follows. We represent $h \in \mathcal{M}(N_{g-1}, p_0)$ by a diffeomorphism h over N_g such that $h(U) = U$ and $h|_U = id_U$. We construct N_g

from $N_{g-1} - \text{int } U$ by attaching a Möbius band along ∂U . Here we assume that this Möbius band is the i -th band on N_g . We extend $h|_{N - \text{int } U}$ to a diffeomorphism $\varphi(h)$ over N_g constructed as above by the identity on the Möbius band. The homomorphism $\psi = \varphi \circ j$ is called a crosscap pushing map.

Every short-leg Y-homeomorphism $Y_{x_i,a}$ is in $\psi(\pi_1(N_{g-1}, p_0))$, $\pi_1(N_{g-1}, p_0)$ is generated by the loops $l_{i,j}$'s indicated in Figure 8, and $\psi(l_{i,j}) = Y_{i,j}$, hence $Y_{x_i,a}$ is a product of $Y_{i,j}$'s. \square

Let G_g be the subgroup of $\mathcal{M}(N_g)$ generated by $\mathcal{YS}_g, t_{a_1}^2, \dots, t_{a_{g-1}}^2, t_{c_1}^2, \dots, t_{c_{g-3}}^2, t_{d_1}, \dots, t_{d_{g-2}}, t_{a_1}t_{a_3}t_{c_1}, \dots, t_{a_{g-3}}t_{a_{g-1}}t_{c_{g-3}}$. We have already shown that $G_g \subset \mathcal{N}_g(q_{os})$, therefore, what we should show is $\mathcal{N}_g(q_{os}) \subset G_g$. Two Y-homeomorphisms Y_1 and Y_2 are G_g -equivalent if there is an element ϕ of G_g such that $\phi Y_1 \phi^{-1} = Y_2$. We remark that if $Y_1 = Y_{m,a}$ and $Y_2 = \phi Y_1 \phi^{-1}$ then $Y_2 = Y_{\phi(m), \phi(a)}$. We will show:

Lemma 4.4. *Every Y-homeomorphism is a product of Y-homeomorphisms which are G_g -equivalent to short-leg Y-homeomorphisms.*

By Lemmas 4.3 and 4.4, we see that every Y-homeomorphism is an element of G_g . Therefore, by Theorem 3.1, we conclude:

Corollary 4.5. $\Gamma_2(N_g) \subset G_g$. \square

Remark 4.6. While the author was writing this paper, Szepietowski informed the author that he found a finite system of generators for $\Gamma_2(N_g)$. In the next subsection, we introduce his system of generators and prove Lemma 4.4 by using his result. In this subsection, we show Lemma 4.4 by our original proof.

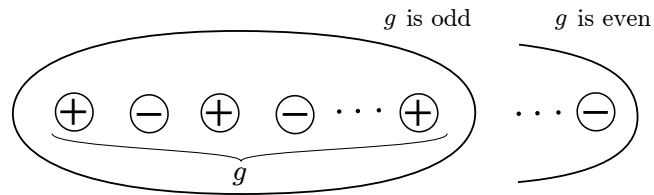


FIGURE 9. Diagram indicating o -standard N_g in S^4 .

As shown in Figure 9, we use the symbol \oplus (resp. \ominus) to indicate the place where the Möbius band are attached such that $q_{os}(x_i) = +1$ (resp. $q_{os}(x_i) = -1$) for the

circle x_i indicated in Figure 1. We denote an element $x = \sum_{i=1}^g \epsilon_i x_i \in H_1(N_g; \mathbb{Z}_2)$, where $\epsilon_i = 0$ or 1 , by a sequence of symbols $+, -, \oplus, \ominus$ of length g with $[\cdot]$ which are settled by the rule: if $\epsilon_{2i-1} = 0$ then the $2i-1$ -st symbol is $+$, if $\epsilon_{2i-1} = 1$ then the $2i-1$ -st symbol is \oplus , if $\epsilon_{2i} = 0$ then the $2i$ -th symbol is $-$, and if $\epsilon_{2i} = 1$ then the $2i$ -th symbol is \ominus . For example, when $g = 7$, we denote an element $x_2 + x_3 + x_6 + x_7$ by $[+ \ominus \oplus - + \ominus \oplus]$. This sequence is called the *r-sequence* associated to x . For the *r-sequence* associated to x , we settle a simple closed curve on N_g by the following rule. For the symbols in this sequence, we put arcs on N_g indicated in the bottom of Figure

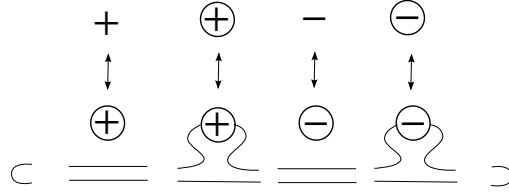


FIGURE 10. Parts of *r*-circles.

10, glue them along the boundaries, and cap by the arc indicated on the left of Figure 10 from the left and by the arc indicated on the right of Figure 10 from the right. We call this circle the *r-circle* associated to x and denote by $R(x)$. For an element $x = \sum_{i=1}^g \epsilon_i x_i \in H_1(N_g; \mathbb{Z}_2)$, where $\epsilon_i = 0$ or 1 , we define $\text{supp}(x) = \{x_i \mid \epsilon_i = 1\}$.

Two simple closed curves c_1 and c_2 on N_g are G_g -equivalent ($c_1 \sim_{G_g} c_2$) if there is an element ϕ of G_g such that $\phi(c_1) = c_2$.

Lemma 4.7. *If $g = 1$, then every r -circle is G_g -equivalent to $R([+])$ or $R([\oplus])$. If $g = 2$, then every r -circle is G_g -equivalent to $R([+-])$, $R([\oplus-])$, $R([+\ominus])$ or $R([\oplus\ominus])$. If $g \geq 3$ is odd, then every r -circle is G_g -equivalent to $R([+-+-\cdots+])$, $R([\oplus-+-\cdots+])$, $R([+\ominus+-\cdots+])$, $R([\oplus\ominus+-\cdots+])$, $R([\oplus-\oplus-\cdots+])$ or $R([\oplus\ominus\oplus\ominus\cdots\oplus])$. If $g \geq 4$ is even, then every r -circle is G_g -equivalent to $R([+-+-\cdots-])$, $R([\oplus-+-\cdots-])$, $R([+\ominus+-\cdots-])$, $R([\oplus\ominus+-\cdots-])$, $R([\oplus-\oplus-\cdots-])$ or $R([\oplus\ominus\oplus\ominus\cdots\ominus])$.*

Proof. If $g = 1$ or 2 , then the conclusion is trivial.

If $g \geq 3$, then

$$\begin{aligned}
(2) \quad & R([\cdots - + \ominus \cdots]) \underset{G_g}{\sim} R([\cdots \ominus + - \cdots]), \\
& R([\cdots + - \oplus \cdots]) \underset{G_g}{\sim} R([\cdots \oplus - + \cdots]), \\
& R([\cdots - \oplus \ominus \cdots]) \underset{G_g}{\sim} R([\cdots \ominus \oplus - \cdots]), \\
& R([\cdots + \ominus \oplus \cdots]) \underset{G_g}{\sim} R([\cdots \oplus \ominus + \cdots]),
\end{aligned}$$

where the left most symbols are the i -th symbol, since $Y_{i+2,i}Y_{i+1,i}t_{d_i}R([\cdots - + \ominus \cdots]) = R([\cdots \ominus + - \cdots])$, $Y_{i+2,i}Y_{i+1,i}t_{d_i}R([\cdots + - \oplus \cdots]) = R([\cdots \oplus - + \cdots])$, $Y_{i+2,i+1}Y_{i+1,i+2}t_{d_i}R([\cdots - \oplus \ominus \cdots]) = R([\cdots \ominus \oplus - \cdots])$ and $Y_{i+2,i+1}Y_{i+1,i+2}t_{d_i}R([\cdots + \ominus \oplus \cdots]) = R([\cdots \oplus \ominus + \cdots])$. Therefore, when $g = 3$, for two cases $R([+ - \oplus])$ and $R([+ \ominus \oplus])$ which are not listed in the statement, we see $R([+ - \oplus]) \underset{G_g}{\sim} R([\oplus - +])$ and $R([+ \ominus \oplus]) \underset{G_g}{\sim} R([\oplus \ominus +])$.

If $g \geq 4$, then

$$\begin{aligned}
(3) \quad & R([\cdots - \oplus \ominus \oplus \cdots]) \underset{G_g}{\sim} R([\cdots - \oplus - + \cdots]), \\
& R([\cdots + \ominus \oplus \ominus \cdots]) \underset{G_g}{\sim} R([\cdots + \ominus + - \cdots]), \\
& R([\cdots \ominus \oplus \ominus + \cdots]) \underset{G_g}{\sim} R([\cdots - + \ominus + \cdots]), \\
& R([\cdots \oplus \ominus \oplus - \cdots]) \underset{G_g}{\sim} R([\cdots + - \oplus - \cdots]), \\
& R([\cdots - \oplus - \oplus \cdots]) \underset{G_g}{\sim} R([\cdots \ominus + \ominus + \cdots]), \\
& R([\cdots + \ominus + \ominus \cdots]) \underset{G_g}{\sim} R([\cdots \oplus - \oplus - \cdots]),
\end{aligned}$$

where the left most symbols are the i -th symbol, since

$$\begin{aligned}
& Y_{i+3,i+1}Y_{i+2,i+1}t_{a_i}^{-2}(t_{a_i}t_{a_{i+2}}t_{c_i})R([\cdots - \oplus \ominus \oplus \cdots]) = R([\cdots - \oplus - + \cdots]), \\
& Y_{i+3,i+1}Y_{i+2,i+1}t_{a_i}^{-2}(t_{a_i}t_{a_{i+2}}t_{c_i})R([\cdots + \ominus \oplus \ominus \cdots]) = R([\cdots + \ominus + - \cdots]), \\
& Y_{i,i+2}Y_{i+1,i+2}t_{a_{i+2}}^2(t_{a_i}t_{a_{i+2}}t_{c_i})^{-1}R([\cdots \ominus \oplus \ominus + \cdots]) = R([\cdots - + \ominus + \cdots]), \\
& Y_{i,i+2}Y_{i+1,i+2}t_{a_{i+2}}^2(t_{a_i}t_{a_{i+2}}t_{c_i})^{-1}R([\cdots \oplus \ominus \oplus - \cdots]) = R([\cdots + - \oplus - \cdots]), \\
& Y_{i+1,i}(t_{a_i}t_{a_{i+2}}t_{c_i})Y_{i+2,i+3}^{-1}R([\cdots - \oplus - \oplus \cdots]) = R([\cdots \ominus + \ominus + \cdots]), \\
& Y_{i+1,i}(t_{a_i}t_{a_{i+2}}t_{c_i})Y_{i+2,i+3}^{-1}R([\cdots + \ominus + \ominus \cdots]) = R([\cdots \oplus - \oplus - \cdots]).
\end{aligned}$$

When $g \geq 4$, we get our conclusion by the induction on g and G_g -equivalences (2) and (3). \square

If the complement of an M-circle m is orientable, then any circle intersecting m transversely in one point is an M-circle. Therefore the leg of every Y-homeomorphism is an M-circle whose complement is non-orientable. Every element of G_g preserves q_{os} , the r-circles $R([+ - \cdots \pm])$, $R([\oplus - \oplus - \cdots \pm])$ and $R([\oplus \ominus + \cdots \pm])$ are A-circles, and the complements of $R([\oplus \ominus \cdots \oplus])$ and $R([\oplus \ominus \cdots \ominus])$ are orientable, hence we see:

Corollary 4.8. *If an r-circle $R(x)$ is a leg of a Y-homeomorphism, then $R(x)$ is G_g -equivalent to $R([\oplus - + \cdots])$ or $R([+ \ominus + \cdots])$. \square*

By investigating the action of Chillingworth's generators for $\mathcal{M}(N_g)$ on legs of Y-homeomorphisms, we see:

Lemma 4.9. *Every Y-homeomorphism is a product of Y-homeomorphisms whose legs are r-circles.*

Proof. Since $\{x_1, \dots, x_g\}$ are r-circles, $Y_{i,j}$ is a Y-homeomorphism whose leg is an r-circle. For every Y-homeomorphism $Y_{m,a}$, there is an r-circle s and an element $\phi \in \mathcal{M}(N_g)$ such that $\phi(s) = m$, that is, $Y_{m,a} = \phi Y_{s,\phi^{-1}(a)} \phi^{-1}$. Therefore, by Lemma 3.4, it suffices to show that there are $\phi_i, \phi'_i, \psi_i, \psi'_i \in \mathcal{YS}_g$ and r-circles s_i, s'_i, t_i, t'_i such that $t_{a_i}(s) = \phi_i(s_i)$, $t_{a_i}^{-1}(s) = \phi'_i(s'_i)$, $t_{c_i}(s) = \psi_i(t_i)$ and $t_{c_i}^{-1}(s) = \psi'_i(t'_i)$ for every r-circle s . As observed in the proof of [14, Lemma 3.1], Y_{x_i, a_i} preserves a_i and exchanges the sides of a_i , hence $t_{a_i} = Y_{x_i, a_i} t_{a_i}^{-1} Y_{x_i, a_i}^{-1}$, therefore $t_{a_i}^2 = t_{a_i} Y_{x_i, a_i} t_{a_i}^{-1} Y_{x_i, a_i}^{-1} = Y_{t_{a_i}(x_i), a_i} Y_{x_i, a_i}^{-1} = Y_{i+1, i} Y_{i, i+1}^{-1}$. By the same way as above, we see that $t_{c_i} = Y_{x_i, c_i} t_{c_i}^{-1} Y_{x_i, c_i}^{-1}$, therefore $t_{c_i}^2 = t_{c_i} Y_{x_i, c_i} t_{c_i}^{-1} Y_{x_i, c_i}^{-1} = Y_{t_{c_i}(x_i), c_i} Y_{x_i, c_i}^{-1}$. Since $Y_{g, i+3} \cdots Y_{i+4, i+3} Y_{i, i+1} \cdots Y_{1, i+1} t_{c_i}(x_i)$ is isotopic to $R(x_{i+1} + x_{i+2} + x_{i+3})$, $t_{c_i}^2$ is a product of Y-homeomorphisms whose legs are r-circles. From the above observation, it suffices to show that one of $t_{a_i}(s) = \phi_i(s_i)$, $t_{a_i}^{-1}(s) = \phi'_i(s'_i)$, and one of $t_{c_i}(s) = \psi_i(t_i)$, $t_{c_i}^{-1}(s) = \psi'_i(t'_i)$.

Since a_i does not intersect $R(x)$ such that $\text{supp}(x) \cap \{x_i, x_{i+1}\} = \emptyset$, we only consider the action of t_{a_i} on $R(x)$ such that $\text{supp}(x) \cap \{x_i, x_{i+1}\} \neq \emptyset$. When we consider the action of t_{a_i} and Y-homeomorphisms, we do not need to take care of the sign on the Möbius bands. Hence, in symbols of r-sequences, we change $+$ and $-$ into \times , and \oplus and \ominus into \otimes . There are 3 cases to consider: $R([\cdots \otimes \times \cdots])$, $R([\cdots \times \otimes \cdots])$, and $R([\cdots \otimes \otimes \cdots])$, where the i -th and $i+1$ -st symbols are indicated. The 3-rd

r-circle does not intersect a_i , hence we ignore this. By drawing figures of r-circles, we see: $Y_{i,i+1}(t_{a_i}^{-1}(R([\cdots \otimes \times \cdots]))) = R([\cdots \times \otimes \cdots])$ and $Y_{i+1,i}(t_{a_i}(R([\cdots \times \otimes \cdots]))) = R([\cdots \otimes \times \cdots])$.

By the same reasons as in the previous paragraph, it suffice to consider the action of t_{c_i} on $R(x)$ such that $\text{supp}(x) \cap \{x_i, x_{i+1}, x_{i+2}, x_{i+3}\} \neq \emptyset$, and, in symbols of r-sequences, we change $+$ and $-$ into \times , and \oplus and \ominus into \otimes . There are 15 cases to consider: (1) $R([\cdots \otimes \times \times \times \cdots])$, (2) $R([\cdots \times \otimes \times \times \cdots])$, (3) $R([\cdots \otimes \otimes \times \times \cdots])$, (4) $R([\cdots \times \times \otimes \times \cdots])$, (5) $R([\cdots \otimes \times \otimes \times \cdots])$, (6) $R([\cdots \times \otimes \otimes \times \cdots])$, (7) $R([\cdots \otimes \otimes \otimes \times \cdots])$, (8) $R([\cdots \times \times \times \otimes \cdots])$, (9) $R([\cdots \otimes \times \times \otimes \cdots])$, (10) $R([\cdots \times \times \times \otimes \cdots])$, (11) $R([\cdots \otimes \otimes \times \otimes \cdots])$, (12) $R([\cdots \times \times \otimes \otimes \cdots])$, (13) $R([\cdots \otimes \times \otimes \otimes \cdots])$, (14) $R([\cdots \times \otimes \otimes \otimes \cdots])$, (15) $R([\cdots \otimes \otimes \otimes \otimes \cdots])$, where the i -th, $i+1$ -st, $i+2$ -nd and $i+3$ -rd symbols are indicated. Since (3) (6) (12) and (15) do not intersect c_i , t_{c_i} does not change these r-circles. By drawing figures of r-circles, we see:

- (1) $Y_{i,i+1}Y_{i+2,i+3}Y_{i+1,i+3}Y_{i+1,i+2}^{-1}(t_{c_i}^{-1}(R([\cdots \otimes \times \times \times \cdots]))) = R([\cdots \times \otimes \otimes \otimes \cdots])$,
- (2) $Y_{i+1,i}Y_{i+2,i+3}(t_{c_i}(R([\cdots \times \otimes \times \times \cdots]))) = R([\cdots \otimes \times \otimes \otimes \cdots])$,
- (4) $Y_{i+2,i+3}Y_{i+1,i}(t_{c_i}^{-1}(R([\cdots \times \times \otimes \times \cdots]))) = R([\cdots \otimes \otimes \times \otimes \cdots])$,
- (5) $Y_{i+1,i+2}Y_{i,i+2}Y_{i+3,i+2}^{-1}Y_{i+2,i+3}^{-1}Y_{i,i+3}^{-1}(t_{c_i}^{-1}(R([\cdots \otimes \times \otimes \times \cdots]))) = R([\cdots \otimes \times \otimes \times \cdots])$,
- (7) $Y_{i,i+3}Y_{i+1,i+3}Y_{i+2,i+3}(t_{c_i}^{-1}(R([\cdots \otimes \otimes \otimes \times \cdots]))) = R([\cdots \times \times \times \otimes \cdots])$,
- (8) $Y_{i+3,i+2}Y_{i+1,i}Y_{i+2,i}Y_{i+2,i+1}^{-1}(t_{c_i}(R([\cdots \times \times \times \otimes \cdots]))) = R([\cdots \otimes \otimes \otimes \times \cdots])$,
- (9) $Y_{i+2,i+3}Y_{i+1,i}Y_{i+3,i}Y_{i+3,i+1}^{-1}Y_{i+3,i+2}Y_{i+2,i+3}Y_{i+1,i+3}Y_{i+1,i+2}^{-1}Y_{i,i+3}Y_{i,i+2}^{-1}Y_{i,i+1}(t_{c_i}^{-1}(R([\cdots \otimes \times \times \otimes \cdots]))) = R([\cdots \otimes \times \times \otimes \cdots])$,
- (10) $Y_{i+2,i+1}Y_{i+3,i+1}Y_{i,i+3}^{-1}Y_{i+1,i}^{-1}Y_{i+3,i}^{-1}(t_{c_i}(R([\cdots \times \otimes \times \otimes \cdots]))) = R([\cdots \times \otimes \times \otimes \cdots])$,
- (11) $Y_{i,i+2}Y_{i+1,i+2}Y_{i+3,i+2}Y_{i+2,i}Y_{i+2,i+1}^{-1}(t_{c_i}(R([\cdots \otimes \otimes \times \otimes \cdots]))) = R([\cdots \times \times \otimes \times \cdots])$,
- (13) $Y_{i+3,i+1}Y_{i+2,i+1}Y_{i,i+1}Y_{i+1,i+3}Y_{i+1,i+2}^{-1}(t_{c_i}^{-1}(R([\cdots \otimes \times \otimes \otimes \cdots]))) = R([\cdots \times \otimes \times \times \cdots])$,
- (14) $Y_{i+3,i}Y_{i+2,i}Y_{i+1,i}(t_{c_i}(R([\cdots \times \otimes \otimes \otimes \cdots]))) = R([\cdots \otimes \times \times \times \cdots])$. \square

Proof of Lemma 4.4. Let $Y_{m,a}$ be a Y-homeomorphism whose leg is an r-circle. By Corollary 4.8, there is an element $\phi \in G_g$ such that $\phi(m) = R([\oplus - + \cdots])$ or $R([+ \ominus + \cdots])$. Therefore $Y_{m,a}$ is G_g -equivalent to a short-leg Y-homeomorphism. By Lemma 4.9, we get our conclusion. \square

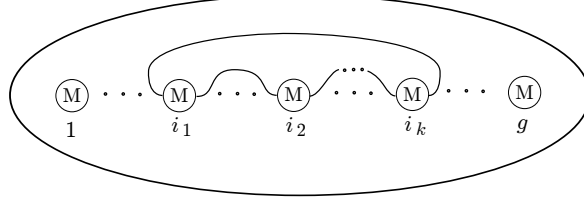


FIGURE 11. The curve α_I for $I = \{i_1, i_2, \dots, i_k\}$.

4.2. Szepietowski's generators for $\Gamma_2(N_g)$. We review the finite system of generators for $\Gamma_2(N_g)$ introduced in [15]. For each non empty subset $I = \{i_1, i_2, \dots, i_k\}$ of $\{1, \dots, g\}$, let α_I be the simple closed curve shown in Figure 11. If $I = \{i\}$, we write α_i instead of $\alpha_{\{i\}}$. Szepietowski proved:

Theorem 4.10. [15, Theorem 3.2] *For $g \geq 4$, $\Gamma_2(N_g)$ is generated by the following elements.*

- 1) $Y_{\alpha_i, \alpha_{\{i,j\}}}$ for $i \neq j$,
- 2) $Y_{\alpha_{\{i,j,k\}}, \alpha_{\{i,j,k,l\}}}$ for $i < j < k < l$.

The group $\Gamma_2(N_3)$ is generated by the elements 1).

We show:

Lemma 4.11. *For arbitrary $i < j < k < l$, $Y_{\alpha_{\{i,j,k\}}, \alpha_{\{i,j,k,l\}}}$ is G_g -equivalent to a short-leg Y -homeomorphism.*

Proof. It suffices to show that, for every $i < j < k$, $\alpha_{\{i,j,k\}}$ is G_g -equivalent to α_1 or α_2 . By drawing figures, we see: when $i > 2$, $Y_{i,i-2}^{-1} Y_{i-1,i-2}^{-1} t_{d_{i-2}} \alpha_{i,j,k} = \alpha_{i-2,j,k}$; when $i < j-2$, $Y_{j,j-2}^{-1} Y_{j-1,j-2}^{-1} t_{d_{j-2}} \alpha_{i,j,k} = \alpha_{i,j-2,k}$; when $j < k-2$, $Y_{k,k-2}^{-1} Y_{k-1,k-2}^{-1} t_{d_{k-2}} \alpha_{i,j,k} = \alpha_{i,j,k-2}$. By applying the above action of G_g on $\alpha_{i,j,k}$, we see that $\alpha_{i,j,k}$ is G_g -equivalent to $\alpha_{1,3,4}$, $\alpha_{1,2,3}$, $\alpha_{2,3,5}$, $\alpha_{2,4,6}$, $\alpha_{1,3,5}$, $\alpha_{1,2,4}$, $\alpha_{2,3,4}$ or $\alpha_{2,4,5}$. By drawing figures of the action of G_g on the above 8 circles, we can check that former 4 circles are G_g -equivalent to α_1 and last 4 circles are G_g -equivalent to α_2 . \square

Every Y -homeomorphism Y is an element of $\Gamma_2(N_g)$, and by Theorem 4.10 and Lemma 4.11, we can express Y as a product of Y -homeomorphisms which are G_g -equivalent to short-leg Y -homeomorphisms. Hence, Lemma 4.4 follows.

4.3. Generators for $\mathcal{O}_g(q_{os})$. If we find a system of generators $\{S_1, \dots, S_k\}$ for $\mathcal{O}_g(q_{os})$ and elements $\sigma_1, \dots, \sigma_k$ of $\mathcal{M}(N_g)$ such that $(\sigma_i)_* = S_i$ in $\text{Aut}(H_1(N_g; \mathbb{Z}_2))$, then, by the short exact sequence (1) and Corollary 4.5, we see that $\mathcal{N}_g(q_{os})$ is generated by $G_g \cup \{\sigma_1, \dots, \sigma_k\}$. Nowik showed in [13, Theorem 3.2.]:

Theorem 4.12. $\mathcal{O}_g(q_{os})$ is generated by the set of elements of the following two forms:

1. T_a for $a \in H_1(N_g; \mathbb{Z}_2)$ with $q_{os}(a) = 2$,
2. $T_a T_b T_{a+b}$ for $a, b \in H_1(N_g; \mathbb{Z}_2)$ with $q_{os}(a) = q_{os}(b) = q_{os}(a+b) = 0$.

Let $\{x_1, \dots, x_g\}$ be the basis of $H_1(N_g; \mathbb{Z}_2)$ which is introduced in Figure 1. We obtain a finite system of generators for $\mathcal{O}_g(q_{os})$ explicitly.

Lemma 4.13. $\mathcal{O}_g(q_{os})$ is generated by

$$(4) \quad T_{x_i+x_{i+2}} \quad (i = 1, \dots, g-2),$$

$$(5) \quad T_{x_i+x_{i+1}} T_{x_{i+2}+x_{i+3}} T_{x_i+x_{i+1}+x_{i+2}+x_{i+3}} \quad (i = 1, \dots, g-3).$$

Proof. We write any element v of $H_1(N_g; \mathbb{Z}_2)$ as $v = x_{i_1} + \dots + x_{i_m}$ such that $i_1 < \dots < i_m$ and call m the length of v , or as $v = (x_{2j_1+1} + \dots + x_{2j_k+1}) \oplus (x_{2j_{k+1}} + \dots + x_{2j_m})$ such that $j_1 \leq \dots \leq j_k$, $j_{k+1} \leq \dots \leq j_m$ and call $(x_{2j_1+1} + \dots + x_{2j_k+1})$ the odd part of v , and $(x_{2j_{k+1}} + \dots + x_{2j_m})$ the even part of v . Two elements v, w of $H_1(N_g; \mathbb{Z}_2)$ is (4)-equivalent $v \stackrel{4}{\sim} w$ if there is a product T of (4) such that $T(v) = w$, and define (5)-equivalence $v \stackrel{5}{\sim} w$ and (4) and (5)-equivalence $v \stackrel{4,5}{\sim} w$ in the same way. We remark that if $v \stackrel{4,5}{\sim} w$ then there is a product T of (4) and (5) such that $T_w = TT_v T$.

Any element (4) acts only on the odd part of v or only on the even part of v . For example, when $i < j < k$,

$$T_{x_{2j-1}+x_{2j+1}}((\dots + x_{2i-1} + x_{2j+1} + x_{2k+1} + \dots) \oplus (\dots)) = \\ ((\dots + x_{2i-1} + x_{2j-1} + x_{2k+1} + \dots) \oplus (\dots)).$$

Therefore, if we define $l_o(v)$ = the length of the odd part of v , and $l_e(v)$ = the length of the even part of v , then $v \stackrel{4}{\sim} (x_1 + x_3 + \dots + x_{2l_o(v)-1}) \oplus (x_2 + x_4 + \dots + x_{2l_e(v)})$. Hence, $v \stackrel{4}{\sim} w$ if and only if $l_o(v) = l_o(w)$ and $l_e(v) = l_e(w)$.

When $p < i, i + 3 < s$, $\{i, i + 1\} = \{q, q'\}$ and $\{i + 2, i + 3\} = \{r, r'\}$, the element (5) acts as follows.

$$\begin{aligned}
& T_{x_i+x_{i+1}} T_{x_{i+2}+x_{i+3}} T_{x_i+x_{i+1}+x_{i+2}+x_{i+3}} (\cdots + x_p + x_q + x_r + x_s + \cdots) \\
&= T_{x_i+x_{i+1}} T_{x_{i+2}+x_{i+3}} (\cdots + x_p + x_q + x_r + x_s + \cdots) \\
&= \cdots + x_p + x_{q'} + x_{r'} + x_s + \cdots, \\
& T_{x_i+x_{i+1}} T_{x_{i+2}+x_{i+3}} T_{x_i+x_{i+1}+x_{i+2}+x_{i+3}} (\cdots + x_p + x_q + x_s + \cdots) \\
&= T_{x_i+x_{i+1}} T_{x_{i+2}+x_{i+3}} (\cdots + x_p + x_{q'} + x_r + x_{r'} + x_s + \cdots) \\
&= \cdots + x_p + x_q + x_r + x_{r'} + x_s + \cdots, \\
& T_{x_i+x_{i+1}} T_{x_{i+2}+x_{i+3}} T_{x_i+x_{i+1}+x_{i+2}+x_{i+3}} (\cdots + x_p + x_r + x_s + \cdots) \\
&= T_{x_i+x_{i+1}} T_{x_{i+2}+x_{i+3}} (\cdots + x_p + x_q + x_{q'} + x_{r'} + x_s + \cdots) \\
&= \cdots + x_p + x_q + x_{q'} + x_r + x_s + \cdots.
\end{aligned}$$

We will show that every element of the first form in Theorem 4.12 is a product of (4) and (5). Let a be an element of $H_1(N_g; \mathbb{Z}_2)$ such that $q_{os}(a) = 2$. Then, $2 \equiv l_o(a) - l_e(a) \pmod{4}$. Therefore, there are two cases $l_o(a) = l_e(a) + 4t + 2$ or $l_e(a) = l_o(a) + 4t + 2$ ($t \in \mathbb{Z}_{\geq 0}$). For the first case,

$$\begin{aligned}
a &\stackrel{4}{\sim} (x_1 + x_3) + (x_4 + x_5) + \cdots + (x_{2i} + x_{2i+1}) + (x_l + x_{l+2} + x_{l+4} + x_{l+6}) + \cdots \\
&\quad \cdots + (x_m + x_{m+2} + x_{m+4} + x_{m+6}).
\end{aligned}$$

For the second case,

$$\begin{aligned}
a &\stackrel{4}{\sim} (x_2 + x_4) + (x_5 + x_6) + \cdots + (x_{2i+1} + x_{2i+2}) + (x_l + x_{l+2} + x_{l+4} + x_{l+6}) + \cdots \\
&\quad \cdots + (x_m + x_{m+2} + x_{m+4} + x_{m+6}).
\end{aligned}$$

We see

$$\begin{aligned}
& (x_2 + x_4) + (x_5 + x_6) + \cdots + (x_{2i+1} + x_{2i+2}) + (x_l + x_{l+2} + x_{l+4} + x_{l+6}) + \cdots \\
& \quad \cdots + (x_m + x_{m+2} + x_{m+4} + x_{m+6}) \stackrel{5}{\sim} \\
& (x_1 + x_3) + (x_5 + x_6) + \cdots + (x_{2i+1} + x_{2i+2}) + (x_l + x_{l+2} + x_{l+4} + x_{l+6}) + \cdots \\
& \quad \cdots + (x_m + x_{m+2} + x_{m+4} + x_{m+6}) \stackrel{4}{\sim} \\
& (x_1 + x_3) + (x_4 + x_5) + \cdots + (x_{2i} + x_{2i+1}) + (x_l + x_{l+2} + x_{l+4} + x_{l+6}) + \cdots \\
& \quad \cdots + (x_m + x_{m+2} + x_{m+4} + x_{m+6}),
\end{aligned}$$

where $\overset{5}{\sim}$ is by $T_{x_1+x_2}T_{x_3+x_4}T_{x_1+x_2+x_3+x_4}$. Therefore, it suffices to consider the first case. We see $(x_l + x_{l+2} + x_{l+4} + x_{l+6}) \overset{5}{\sim} (x_l + x_{l+2} + x_{l+3} + x_{l+5}) \overset{5}{\sim} (x_l + x_{l+5}) \overset{4}{\sim} (x_l + x_{l+1})$, where the first $\overset{5}{\sim}$ is by $T_{x_{l+3}+x_{l+4}}T_{x_{l+5}+x_{l+6}}T_{x_{l+3}+x_{l+4}+x_{l+5}+x_{l+6}}$, and the second $\overset{5}{\sim}$ is by $T_{x_{l+2}+x_{l+3}}T_{x_{l+4}+x_{l+5}}T_{x_{l+2}+x_{l+3}+x_{l+4}+x_{l+5}}$. Hence, $a \overset{4,5}{\sim} (x_1 + x_3) + (x_4 + x_5) + \cdots + (x_{2n} + x_{2n+1})$. Furthermore, $(x_1 + x_3) + (x_4 + x_5) + (x_6 + x_7) + \cdots + (x_{2n} + x_{2n+1}) = (x_1+x_3+x_4)+x_5+(x_6+x_7)+\cdots+(x_{2n}+x_{2n+1}) \overset{5}{\sim} x_1+x_5+(x_6+x_7)+\cdots+(x_{2n}+x_{2n+1}) \overset{4}{\sim} x_1+x_3+(x_4+x_5)+\cdots+(x_{2(n-1)}+x_{2(n-1)+1})$, where $\overset{5}{\sim}$ is by $T_{x_1+x_2}T_{x_3+x_4}T_{x_1+x_2+x_3+x_4}$. Therefore, by the induction on n , we see $a \overset{4,5}{\sim} x_1 + x_3$. Hence T_a is a product of (4) and (5) if $q_{os}(a) = 2$.

We will show that every element of the second form in Theorem 4.12 is a product of (4) and (5). Let a and b be elements of $H_1(N_g; \mathbb{Z}_2)$ such that $q_{os}(a) = q_{os}(b) = q_{os}(a+b) = 0$. Then $0 = q_{os}(a+b) = q_{os}(a) + q_{os}(b) + (a \cdot b)_2 = (a \cdot b)_2$, $0 = q_{os}(2a) = q_{os}(a) + q_{os}(a) + (a \cdot a)_2 = (a \cdot a)_2$, by the same reason, $0 = (b \cdot b)_2$, hence $(a + b \cdot a)_2 = (a + b \cdot b)_2 = 0$. Therefore, T_a , T_b and T_{a+b} commute each other. For the pairs $[a_1, b_1]$ and $[a_2, b_2]$ of elements of $H_1(N_g; \mathbb{Z}_2)$ which satisfies $q_{os}(a_i) = q_{os}(b_i) = q_{os}(a_i + b_i) = 0$ ($i = 1, 2$), we define the equivalence $[a_1, b_1] \overset{4}{\sim} [a_2, b_2]$ if there is a product T of (4) such that $T(a_1) = a_2$ and $T(b_1) = b_2$. The equivalences $[a_1, b_1] \overset{5}{\sim} [a_2, b_2]$ and $[a_1, b_1] \overset{4,5}{\sim} [a_2, b_2]$ are defined in the same way.

Let a, b be elements of $H_1(N_g; \mathbb{Z}_2)$ such that $q_{os}(a) = q_{os}(b) = q_{os}(a+b) = 0$. By the same argument applied for elements of the first form in Theorem 4.12, we see $a \overset{4,5}{\sim} (x_1 + x_2) + \cdots + (x_{2n-1} + x_{2n})$.

If $2n \neq g$, then $(x_1 + x_2) + \cdots + (x_{2n-3} + x_{2n-2}) + (x_{2n-1} + x_{2n}) = (x_1 + x_2) + \cdots + x_{2n-3} + (x_{2n-2} + x_{2n-1} + x_{2n}) \overset{5}{\sim} (x_1 + x_2) + \cdots + x_{2n-3} + x_{2n} \overset{4}{\sim} (x_1 + x_2) + \cdots + x_{2n-3} + x_{2n-2} = (x_1 + x_2) + \cdots + (x_{2(n-1)-1} + x_{2(n-1)})$, where $\overset{5}{\sim}$ is by $T_{x_{2n-2}+x_{2n-1}}T_{x_{2n}+x_{2n+1}}T_{x_{2n-2}+x_{2n-1}+x_{2n}+x_{2n+1}}$. Therefore, by the induction on n , we see $a \overset{4,5}{\sim} x_1 + x_2$. Hence, $[a, b] \overset{4,5}{\sim} [x_1+x_2, b']$. Since $0 = (x_1+x_2 \cdot b')_2$, $b' = x_1+x_2+x_l+\cdots+x_m$ or $b' = x_l+\cdots+x_m$ where $l \geq 3$. For these 2 cases, $T_{x_1+x_2}T_{b'}T_{x_1+x_2+b'} = T_{x_1+x_2}T_{x_1+x_2+b'}T_{b'}$ are the same. So, we may suppose $b' = x_l + \cdots + x_m$. We see $[a, b] \overset{4,5}{\sim} [x_1 + x_2, (x_3 + x_4) + \cdots + (x_{2k-1} + x_{2k})]$, by applying (4) and (5) whose transvections are about v such that $\text{supp}(v)$ contains neither x_1 nor x_2 . If $2k \neq g$, by applying to b the same argument

as to a , we see $[a, b] \stackrel{4,5}{\sim} [x_1 + x_2, x_3 + x_4]$. If $2k = g$, then $T_a T_b T_{a+b}$ is equal to $T_{x_1+x_2} T_{x_3+\dots+x_g} T_{x_1+x_2+x_3+\dots+x_g}$. In the last paragraph of this proof, we show that $T_{x_1+x_2} T_{x_3+\dots+x_g} T_{x_1+x_2+x_3+\dots+x_g}$ is a product of (4) and (5).

If $2n = g$, then $(v \cdot (x_1 + x_2) + \dots + (x_{2n-1} + x_{2n}))_2 = 0$ for every $v \in H_1(N_g; \mathbb{Z}_2)$ used for the transvections in (4) and (5). Therefore, we see $[a, b] \stackrel{4,5}{\sim} [(x_1 + x_2) + \dots + (x_{g-1} + x_g), (x_{2i+1} + x_{2i+2}) + \dots + (x_{g-1} + x_g)]$. This means that $T_a T_b T_{a+b} = T_{a+b} T_b T_a$ is conjugate to $T_{(x_1+x_2)+\dots+(x_{2i-1}+x_{2i})} T_{(x_{2i+1}+x_{2i+2})+\dots+(x_{g-1}+x_g)} T_{(x_1+x_2)+\dots+(x_{g-1}+x_g)}$ by (4) and (5). Moreover, $[((x_1 + x_2) + \dots + (x_{2i-3} + x_{2i-2}) + (x_{2i-1} + x_{2i})), (x_{2i+1} + x_{2i+2}) + \dots + (x_{g-1} + x_g)] \stackrel{5}{\sim} [(x_1 + x_2) + \dots + x_{2i-3} + x_{2i}, x_{2i-2} + x_{2i-1} + (x_{2i+1} + x_{2i+2}) + \dots + (x_{g-1} + x_g)] \stackrel{4}{\sim} [(x_1 + x_2) + \dots + (x_{2i-3} + x_{2i-2}), (x_{2i-1} + x_{2i}) + (x_{2i+1} + x_{2i+2}) + \dots + (x_{g-1} + x_g)]$, where $\stackrel{5}{\sim}$ is by $T_{x_{2i-2}+x_{2i-1}} T_{x_{2i}+x_{2i+1}} T_{x_{2i-2}+x_{2i-1}+x_{2i}+x_{2i+1}}$, and $\stackrel{4}{\sim}$ is by $T_{x_{2i-2}+x_{2i}}$. By repeatedly applying the above argument, we see that $T_a T_b T_{a+b}$ is conjugate to $T_{x_1+x_2} T_{x_3+\dots+x_g} T_{x_1+x_2+x_3+\dots+x_g}$ by (4) and (5).

When g is even and $g \geq 6$, by checking the action of transvections on the basis $\{x_1, \dots, x_g\}$ of $H_1(N_g; \mathbb{Z}_2)$, we show $T_{x_1+x_2} T_{x_3+\dots+x_g} T_{x_1+x_2+x_3+\dots+x_g} = T_{x_1+x_2} T_{x_3+x_4} T_{x_1+x_2+x_3+x_4} \cdot T_{x_1+x_2} T_{x_5+\dots+x_g} T_{x_1+x_2+x_5+\dots+x_g}$. Since $[x_1 + x_2, x_5 + \dots + x_g] \stackrel{4}{\sim} [x_1 + x_2, x_3 + \dots + x_{g-2}]$, we see that $T_{x_1+x_2} T_{x_5+\dots+x_g} T_{x_1+x_2+x_5+\dots+x_g}$ is a product of (4) and (5) by using the argument for the case where $2n(=g-2) \neq g$. \square

Since $(t_{d_i})_* = T_{x_i+x_{i+2}}$, $(t_{a_i} t_{a_{i+2}} t_{c_i})_* = T_{x_i+x_{i+1}} T_{x_{i+2}+x_{i+3}} T_{x_i+x_{i+1}+x_{i+2}+x_{i+3}}$, and $t_{d_i}, t_{a_i} t_{a_{i+2}} t_{c_i} \in G_g$, we see $G_g = \mathcal{N}_g(q_{os})$, hence Theorem 4.1 follows.

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